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Scaling in magnetohydrodynamic convection at high Rayleigh number

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The theory of Grossmann and Lohse [J. Fluid Mech. **407**, 27 (2000)] is extended to include the effect of a magnetic field on convection of an electrically conducting fluid. Different scaling laws are obtained depending on whether the bulk or the boundary layers make the major contribution to the dissipation. Scalings are obtained for both weak and strong magnetic fields. The predictions are shown to be in better agreement with experimental data than earlier theoretical models.

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A horizontal layer of fluid, when heated from below, becomes unstable when the Rayleigh number Ra exceeds a critical value Ra_c. Beyond this value convection sets in leading to enhancement in heat transport, which is measured by the Nusselt number Nu. For $\mathrm{Ra} \gg \mathrm{Ra}_c$ the convective motion becomes turbulent and there has been considerable interest in trying to predict the dependence of Nu on Ra for very high values of Ra. Experimental studies [1–11] show a power-law dependence $\mathrm{Nu} \sim \mathrm{Ra}^{\beta}$ with values of β between 1/4 and 1/2. These experimental observations have been compared with available theoretical models [12–23]. One of the most comprehensive of these is the Grossmann and Lohse (GL) model [20], further extended in [21–23], which considers different regimes and seems to explain well the 1/4, 2/7, and 1/3 power laws observed at relatively low, intermediate, and very high values of Ra. In a more recent study, Grossmann and Lohse [24] considered the role of plumes and used a decomposition for the thermal dissipation rate into the plume and the background contributions instead of that into the boundary layer and the bulk contributions [20]. This was motivated by numerical studies of high Ra convection, although there was no disagreement between the earlier theory [20] and experiments as far as the scaling laws were concerned. Further, even with the new theory the scaling laws remained unchanged. At present there is not enough numerical or experimental data to decide whether plumes play a significant role in high-Ra magnetohydrodynamic convection.

One reason for interest in high-Ra convection is that in astrophysics and geophysics we often have convection occurring at extremely high values of Ra. Usually a magnetic field is present and it is known that a magnetic field can suppress convection when the fluid is electrically conducting. Therefore, it would be of interest to study the effect of a magnetic field on high-Ra convection of an electrically conducting fluid. Some theoretical and experimental work on this has been reported [25–28]; however, there is still not very good agreement between theoretical models and experimental data. The numerical results show very good agreement with the experimental data but these computations are time consuming and, therefore, there is need for simple theoretical models to explain the experimental results. With that aim, in this study, we generalize the GL model [20] to include the effect of an imposed vertical magnetic field.

The governing equations for an electrically conducting fluid in a magnetic field, using the Boussinesq approximation, are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \mathbf{J} \times \mathbf{B} + g\alpha\theta \mathbf{k} + \nu \nabla^2 \mathbf{u}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} - \frac{1}{\sigma} \nabla \times \mathbf{J}, \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4)$$

$$\mu \mathbf{J} = \nabla \times \mathbf{B}, \quad (5)$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \kappa \nabla^2 \theta, \quad (6)$$

where $\mathbf{u}$ is the fluid velocity, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current density, θ and p are the temperature and pressure, ρ, α, ν, σ, and κ are the density, coefficient of thermal expansion, kinematic viscosity, electrical conductivity, and thermal diffusivity of the fluid, g is the acceleration due to gravity, and μ is the permeability, assumed to be that of vacuum. SI units are used so that the units of μ, B, J, and σ are T m/A, T, A/m^2, and ohm^{-1}/m. Here temperature is measured with respect to a reference temperature and pressure with respect to the hydrostatic pressure field corresponding to this reference temperature. We have used a Cartesian coordinate system (x,y,z) with unit vectors $(\mathbf{i},\mathbf{j},\mathbf{k})$ along the coordinate axes. The fluid is assumed to occupy the region $-\infty < x,y < \infty$, $0 < z < L$. Gravity is in the direction $-\mathbf{k}$. We assume that the boundaries at $z=0$ and L are rigid and have infinite thermal and electrical conductivity. Each of the boundaries is assumed to be maintained at constant temperature. Choosing the temperature of the upper boundary as the reference temperature and assuming that a temperature difference Δ is imposed across the layer to drive convection, the appropriate boundary conditions are [29]

$$\mathbf{u} = \mathbf{0} \quad \text{at } z = 0 \text{ and } L, \quad (7)$$

$$\theta = \Delta \quad \text{at } z = 0, \quad \theta = 0 \quad \text{at } z = L, \quad (8)$$

$$B_z = B_0, \quad J_x = J_y = 0 \quad \text{at } z = 0 \text{ and } L, \quad (9)$$

where B_0 is the imposed uniform vertical magnetic field.

Following Ref. [20] the starting point for our analysis is the dissipation rates

$$\epsilon_u = \nu \langle |\nabla \mathbf{u}|^2 \rangle, \quad \epsilon_J = \frac{1}{\rho\sigma} \langle |\mathbf{J}|^2 \rangle, \quad \epsilon_\theta = \kappa \langle |\nabla \theta|^2 \rangle, \quad (10)$$

where angular brackets denote volume averages over the fluid layer. Here ϵ_u and ϵ_θ are same as in Ref. [20] while ϵ_J

is the averaged Ohmic dissipation rate per unit mass. We assume a stationary state where all volume averages are time independent. Using the governing equations these can be shown to obey certain rigorous relations. Scalar-multiplying Eq. (1) by $\mathbf{u}$ and averaging over the fluid layer, we obtain

$$0 = \frac{1}{\rho\mu}\langle \mathbf{u}\cdot(\mathbf{B}\cdot\nabla\mathbf{B})\rangle + \alpha g\langle w\theta\rangle - \nu\langle|\nabla\mathbf{u}|^2\rangle, \qquad (11)$$

where w is the z component of $\mathbf{u}$. Again scalar-multiplying Eq. (3) by $\mathbf{B}$ and averaging over the fluid layer, it can be readily shown using Eq. (9) that

$$0 = \langle \mathbf{B}\cdot(\mathbf{B}\cdot\nabla\mathbf{u})\rangle - \frac{\mu}{\sigma}\langle|\mathbf{J}|^2\rangle. \qquad (12)$$

From Eqs. (11) and (12) we obtain [25]

$$\epsilon_u + \epsilon_J = \alpha g\langle w\theta\rangle. \qquad (13)$$

This equation states that the sum of the viscous and Ohmic dissipation rates is equal to the rate of energy released due to buoyancy force. From Eq. (6) it can be shown that [16]

$$\langle w\theta\rangle = \frac{\kappa\Delta}{L}(\mathrm{Nu}-1), \qquad (14)$$

where the Nusselt number is defined by $\mathrm{Nu}=(L/\Delta)\times(-d\bar{\theta}/dz)_{z=0}$, the overbar on θ denotes an average over a horizontal plane, and it is assumed that horizontal averages again are time independent. From Eqs. (13) and (14) we obtain

$$\epsilon_u + \epsilon_J = \frac{\nu^3}{L^4}\frac{\mathrm{Ra}}{\mathrm{Pr}^2}(\mathrm{Nu}-1), \qquad (15)$$

where the Rayleigh number $\mathrm{Ra}=g\alpha\Delta L^3/\kappa\nu$ and the Prandtl number $\mathrm{Pr}=\nu/\kappa$. Multiplying Eq. (6) by θ and averaging over the fluid layer, we obtain [16]

$$\kappa\langle|\nabla\theta|^2\rangle = -\frac{\kappa\Delta}{L}\left(\frac{d\bar{\theta}}{dz}\right)_{z=0}. \qquad (16)$$

It readily follows that

$$\epsilon_\theta = \kappa\frac{\Delta^2}{L^2}\mathrm{Nu}. \qquad (17)$$

Dissipation takes place both in the bulk and in the boundary layers which form near the walls. In the absence of a magnetic field a hydrodynamic boundary layer of thickness λ_u and a thermal boundary layer of thickness λ_θ form, with

$$\lambda_u \sim L/\mathrm{Re}^{1/2}, \quad \lambda_\theta \sim L/\mathrm{Nu}, \qquad (18)$$

where it has been assumed that flow inside the boundary layers is laminar. In the presence of a strong magnetic field both the velocity and the magnetic field vary rapidly inside a Hartmann boundary layer of thickness λ_H [30], while the temperature varies rapidly in the thermal boundary layer of thickness λ_θ. Again assuming laminar flow inside the boundary layers it can be shown that

$$\lambda_H \sim L/Q^{1/2}, \quad \lambda_\theta \sim L/\mathrm{Nu}, \qquad (19)$$

where $Q=B_0^2\sigma L^2/\rho\nu$ is the Chandrasekhar number. The derivation of λ_H is similar to that for λ_u and involves ordering of the terms in the governing equations. Following Ref. [20] we decompose the globally averaged dissipation rates into their boundary layer (BL) and bulk contributions

$$\epsilon_u = \epsilon_{u,\mathrm{BL}} + \epsilon_{u,\mathrm{bulk}}, \qquad (20)$$

with similar expressions for ϵ_J and ϵ_θ.

In the bulk Grossmann and Lohse [20] assumed that there is a balance between the dissipation and the large-scale convective term. In the absence of a magnetic field this leads to

$$\epsilon_{u,\mathrm{bulk}} \sim \frac{U^3}{L} \sim \frac{\nu^3}{L^4}\mathrm{Re}^3, \qquad (21)$$

where the Reynolds number is defined by $\mathrm{Re}=UL/\nu$. Here U is the mean large-scale velocity near the boundaries of the cell, the "thermal wind" first observed by Krishnamurti and Howard [31]. When a magnetic field is present, we assume that in the magnetic induction equation the balance is between the dissipation term, which can also be written as $\eta\nabla^2\mathbf{B}$, where the magnetic diffusivity $\eta=1/\mu\sigma$, and the term $\mathbf{B}\cdot\nabla\mathbf{u}$. Assuming that the induced magnetic field is $\sim\Delta B$, this requires $\eta\Delta B/L^2\sim B_0U/L$. Consequently

$$\epsilon_{J,\mathrm{bulk}} \sim \frac{1}{\rho\sigma}\frac{(\Delta B)^2}{\mu^2L^2} \sim \frac{B_0^2U^2}{\rho\mu\eta} = \frac{\nu^3}{L^4}\mathrm{Re}^3\frac{Q}{\mathrm{Re}}. \qquad (22)$$

Therefore, when $Q\gg\mathrm{Re}$, $\epsilon_{u,\mathrm{bulk}}\ll\epsilon_{J,\mathrm{bulk}}$ and consequently $\epsilon_{u,\mathrm{bulk}}$ can be neglected compared to $\epsilon_{J,\mathrm{bulk}}$. The ordering for ΔB would seem to lead to a contradiction in the momentum equation since

$$\left|\frac{1}{\rho}\mathbf{J}\times\mathbf{B}\right| \sim \frac{B_0\Delta B}{\rho\mu L} \sim \frac{B_0^2U}{\rho\mu\eta} \sim \frac{U^2}{L}\frac{Q}{\mathrm{Re}},$$

$$|\mathbf{u}\cdot\nabla\mathbf{u}| \sim \frac{U^2}{L}.$$

For $Q\gg\mathrm{Re}$ it is not clear what balances the $\mathbf{J}\times\mathbf{B}$ force. However, it is known [32,33] that in strongly magnetized plasmas the magnetic field relaxes to a force-free state so that $|\mathbf{J}\times\mathbf{B}|\ll|\mathbf{J}||\mathbf{B}|$. Then the $|\mathbf{J}\times\mathbf{B}|$ term can be small enough to be balanced by the other terms in the momentum equation. In the temperature equation again we assume a balance between the dissipation and the large-scale convective term. When $\lambda_u,\lambda_H<\lambda_\theta$, the appropriate velocity scale is U and we have

$$\epsilon_{\theta,\mathrm{bulk}} \sim \frac{U\Delta^2}{L} = \kappa\frac{\Delta^2}{L^2}\mathrm{Pr}\,\mathrm{Re}. \qquad (23)$$

In the absence of a strong magnetic field if $\lambda_u>\lambda_\theta$, where the thermal BL meets the bulk, the velocity is $U\lambda_\theta/\lambda_u$ and this provides the appropriate velocity scale. Consequently

$$\epsilon_{\theta,\mathrm{bulk}} \sim \frac{\lambda_\theta}{\lambda_u}\frac{U\Delta^2}{L} = \kappa\frac{\Delta^2}{L^2}\frac{\mathrm{Pr}\,\mathrm{Re}^{3/2}}{\mathrm{Nu}}. \qquad (24)$$

Similarly, in the presence of a strong magnetic field if $\lambda_H>\lambda_\theta$ the appropriate velocity scale is $U\lambda_\theta/\lambda_H$ and consequently

$$\epsilon_{\theta,\mathrm{bulk}} \sim \frac{\lambda_\theta}{\lambda_H}\frac{U\Delta^2}{L} = \kappa\frac{\Delta^2}{L^2}\frac{\mathrm{Pr}\,\mathrm{Re}Q^{1/2}}{\mathrm{Nu}}. \qquad (25)$$

We next derive estimates for dissipation rates in the boundary layers. In the absence of a magnetic field we have [20]

$$\epsilon_{u,\mathrm{BL}} \sim \frac{\nu^3}{L^4}\mathrm{Re}^{5/2}. \quad (26)$$

In the presence of a strong magnetic field we have

$$\epsilon_{u,\mathrm{BL}} \sim \nu \frac{U^2}{\lambda_H^2}\frac{\lambda_H}{L} \sim \frac{\nu^3}{L^4}\mathrm{Re}^2 Q^{1/2}, \quad (27)$$

$$\epsilon_{J,\mathrm{BL}} \sim \frac{1}{\rho\sigma}\frac{(\Delta B)^2}{\mu^2\lambda_H^2}\frac{\lambda_H}{L} \sim \frac{\nu^3}{L^4}\mathrm{Re}^2 Q^{1/2}. \quad (28)$$

Estimating $\epsilon_{\theta,\mathrm{BL}}$ again leads to the same expression as in Eq. (17). So Grossmann and Lohse [20] went back and did an ordering of the terms in Eq. (6) to derive certain relations. Following that procedure, for $\lambda_u, \lambda_H < \lambda_\theta$ we obtain

$$\mathrm{Nu} \sim Re^{1/2}\mathrm{Pr}^{1/2}, \quad (29)$$

while, in the absence of a magnetic field, for $\lambda_u > \lambda_\theta$

$$\mathrm{Nu} \sim Re^{1/2}\mathrm{Pr}^{1/3}, \quad (30)$$

and in the presence of a strong magnetic field, for $\lambda_H > \lambda_\theta$,

$$\mathrm{Nu} \sim \mathrm{Re}^{1/3}\mathrm{Pr}^{1/3}Q^{1/6}. \quad (31)$$

We now derive scaling laws following the procedure of Ref. [20]. As in Ref. [20] we consider the four regimes (I) both $\epsilon_u+\epsilon_J$ and ϵ_θ are dominated by their BL contributions; (II) ϵ_θ is dominated by $\epsilon_{\theta,\mathrm{BL}}$ and $\epsilon_u+\epsilon_J$ is dominated by $\epsilon_{u,\mathrm{bulk}}+\epsilon_{J,\mathrm{bulk}}$; (III) $\epsilon_u+\epsilon_J$ is dominated by $\epsilon_{u,\mathrm{BL}}+\epsilon_{J,\mathrm{BL}}$ and ϵ_θ is dominated by $\epsilon_{\theta,\mathrm{bulk}}$; and (IV) both $\epsilon_u+\epsilon_J$ and ϵ_θ are bulk dominated. Grossmann and Lohse [20] further use subscripts l and u to distinguish the situations $\lambda_u < \lambda_\theta$ and $\lambda_u > \lambda_\theta$ which, in the absence of a magnetic field, correspond to low and high Pr.

For very high Ra the boundary layers are very thin and we expect the dissipation rates to be dominated by contributions from the bulk. Therefore, this is in regime IV and if we assume $\lambda_u, \lambda_H < \lambda_\theta$ it is regime IV_l. Substituting for ϵ_u and ϵ_J from Eqs. (21) and (22) in Eq. (15) and for ϵ_θ from Eq. (23) in Eq. (17) we obtain

$$\frac{\mathrm{Ra}}{\mathrm{Pr}^2}\mathrm{Nu} \sim \mathrm{Re}^3\left(1 + C\frac{Q}{\mathrm{Re}}\right), \quad (32)$$

$$\mathrm{Nu} \sim \mathrm{Pr}\,\mathrm{Re}, \quad (33)$$

where C is a constant that depends on the ratio of Ohmic to viscous dissipation. From Eqs. (32) and (33) it readily follows that

$$\mathrm{Nu} \sim \frac{\mathrm{Ra}^{1/2}\mathrm{Pr}^{1/2}}{(1 + CQ\mathrm{Re}^{-1})^{1/2}}. \quad (34)$$

For $Q=0$ this reduces to $\mathrm{Nu} \sim \mathrm{Ra}^{1/2}\mathrm{Pr}^{1/2}$ as in Ref. [20]. For $Q=0$ we also obtain $\mathrm{Re} \sim \mathrm{Ra}^{1/2}\mathrm{Pr}^{-1/2}$. For small values of Q we can use this expression in the term involving Q to obtain

$$\mathrm{Nu} \sim \frac{\mathrm{Ra}^{1/2}\mathrm{Pr}^{1/2}}{(1 + C_1 Q\mathrm{Ra}^{-1/2}\mathrm{Pr}^{1/2})^{1/2}}, \quad (35)$$

where C_1 is a constant. This shows that Nu decreases with increase in Q, as expected. When a strong magnetic field is present so that $Q/\mathrm{Re} \gg 1$, ϵ_u can be neglected in comparison with ϵ_J, and Eq. (32) can be approximated by

$$\frac{\mathrm{Ra}}{\mathrm{Pr}^2}\mathrm{Nu} \sim \mathrm{Re}^2 Q. \quad (36)$$

From Eqs. (33) and (36) we readily obtain

$$\mathrm{Nu} \sim \frac{\mathrm{Ra}}{Q}. \quad (37)$$

This is identical with the relation derived in Ref. [26] using a local stability criterion for the boundary layer. We now consider regime IV_u where $\lambda_u, \lambda_H > \lambda_\theta$. Equation (32) still remains valid. In the presence of a weak magnetic field, substituting for ϵ_θ from Eq. (24) in Eq. (17), we obtain in place of Eq. (33)

$$\mathrm{Nu} \sim \frac{\mathrm{Pr}\,\mathrm{Re}^{3/2}}{\mathrm{Nu}}. \quad (38)$$

Following a procedure similar to that for regime IV_l, in the presence of a weak magnetic field we obtain

$$\mathrm{Nu} \sim \frac{\mathrm{Ra}^{1/3}}{(1 + C_2 Q\mathrm{Ra}^{-4/9}\mathrm{Pr}^{2/3})^{1/3}}, \quad (39)$$

where C_2 is again a constant. When a strong magnetic field is present λ_H is very small and we do not expect the condition $\lambda_H > \lambda_\theta$, required for the configuration to be in regime IV_u, to be satisfied.

In laboratory experiments using mercury we have $\lambda_u, \lambda_H < \lambda_\theta$. Therefore, as Ra is reduced we expect to go from regime IV_l to regime II_l. In this regime Eq. (32) still holds, together with Eq. (29). From these two equations it can be readily shown that in the presence of a weak magnetic field we have

$$\mathrm{Nu} \sim \frac{\mathrm{Ra}^{1/5}\mathrm{Pr}^{1/5}}{(1 + C_3 Q\mathrm{Ra}^{-2/5}\mathrm{Pr}^{3/5})^{1/5}}, \quad (40)$$

where C_3 is a constant, while in the presence of a strong magnetic field Eq. (32) is replaced by Eq. (36) and we have

$$\mathrm{Nu} \sim \frac{\mathrm{Ra}^{1/3}}{Q^{1/3}}. \quad (41)$$

As we decrease Ra further we expect to go to regime I. Substituting for ϵ_u and ϵ_J from Eqs. (27) and (28) in Eq. (15), we obtain

$$\frac{\mathrm{Ra}}{\mathrm{Pr}^2}\mathrm{Nu} \sim \mathrm{Re}^2 Q^{1/2}. \quad (42)$$

Further, assuming that $\lambda_H < \lambda_\theta$, Eq. (29) is applicable. From these two equations we readily obtain

$$\mathrm{Nu} \sim \frac{\mathrm{Ra}^{1/3}}{Q^{1/6}}. \quad (43)$$

The relative weight of the two dissipation terms in Eq. (15) does introduce an additional parameter. However, in the limiting cases where one of the terms is assumed negligible this additional parameter does not appear. One limiting case is the original GL model [20] where ϵ_J is not present; the other is the strong magnetic field regime where ϵ_u is considered negligible compared to ϵ_J. However, in general, an additional parameter is present as seen in Eqs. (35), (39), and (40). This parameter, like the prefactors in the GL model, will have to be determined from experiments.

Bhattacharjee *et al.* [26] provide the scaling laws $\mathrm{Nu} \sim \mathrm{Ra}/Q$ and $\mathrm{Nu} \sim \mathrm{Ra}^{1/2}/Q^{3/4}$ in two different regimes. The experimental data of Cioni *et al.* [27] show very good agreement with their predictions in the first regime but in the second regime their data show a much weaker dependence on Q, which is better approximated by $\mathrm{Nu} \sim \mathrm{Ra}^{0.43}/Q^{0.25}$. This is quite close to our predictions in regimes I and II. Grossmann and Lohse [20] had also shown that some empirical fits to experimental data can be explained by superposing two scalings. We have derived the scalings in Eqs. (37) and (43) by neglecting the dissipation in the boundary layers and in the bulk. Taking into account that both contributions are present we can use a superposition of these two scalings. Since the power-law exponents for Ra and Q in these two scalings bracket the values 0.43 and −0.25 of the fit to experimental data it appears that a suitable superposition can approximate the experimental data well. However, it should be pointed out that agreement between theory and experiment, in the absence of a magnetic field, was demonstrated using data spanning several decades in Ra. The experimental data for convection in the presence of a magnetic field do not span even one decade in Q while the numerical results reported consist of just five data points. Therefore, more data are required before the question of quantitative agreement can be settled. In the absence of a magnetic field there have been theoretical predictions supported by some experimental observations that the power-law exponent goes up for very high values of Ra. The experimental observations in the presence of a magnetic field show just the opposite trend. However, for very high values of Ra our model predicts a linear increase of Nu with Ra in the presence of a strong magnetic field. Thus the power-law exponent is twice what it is in the absence of a magnetic field, assuming that $\lambda_u, \lambda_H < \lambda_\theta$. A similar trend was obtained by Montgomery [25] who found bounds for Nu which scale as $\mathrm{Ra}^{3/8}$ when the magnetic field is weak but as $\mathrm{Ra}^{3/4}$ in the presence of a strong magnetic field. The bounds also contain a numerical factor which again depends on Ra but even when this is taken into account we expect the effective power-law exponent to be higher when a strong magnetic field is present. Since the experiments of Cioni *et al.* [27] did not go to very high values of Ra it is difficult to say what the trend would be in higher ranges of Ra. One limitation of the present study is that for high values of Ra the boundary layers can become turbulent and this has not been considered in our model. Also since our model assumes fully turbulent flow it cannot predict the scalings just beyond the onset of instability where the experimental findings are explained well by the local stability theory [26]. In conclusion our model shows better agreement with experimental data especially in predicting the Q dependence of Nu but more experimental data and numerical results and further refinement of theory are needed before a firm statement about quantitative agreement can be made. We have also provided scalings that show the effect of a weak magnetic field which has not been done earlier to our knowledge, and these are of practical importance since convection in stars is usually in the kinetic regime [34] where the magnetic field is weak.

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